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Hamiltonian and BRST Formulations of a Two-Dimensional Abelian Higgs Model in the Broken Symmetry Phase

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The Hamiltonian and BRST formulations of an Abelian Higgs model involving the electromagnetic vector gauge field are investigated in one-space, one-time dimension in the broken symmetry phase, where the phase $\phi(x, t)$ of the complex matter field $\Phi(x, t)$ carries the charge degree of freedom of the complex matter field and is, in fact, akin to the Goldstone boson.

1. INTRODUCTION

Quantum electrodynamics (QED) models with a Higgs potential namely, Abelian Higgs models (AHM) involving the vector gauge field $A^\mu (x, t)$ in lower [one-space, one-time (1 + 1) or two-space, one-time (2 + 1)] dimensions have attracted wide interest in recent years [1–8]. These models, involving a Maxwell term which accounts for the kinetic energy of the vector gauge field $A^\mu(x, t)$ [1–6], represent field-theoretic models which could be considered effective theories of Ginsburg–Landau type for superconductivity [6]. These models are in fact relativistic generalizations of Ginsburg–Landau (GL) phenomenological field theory models of superconductivity [5]. Some basics of the AHM in the symmetry phase (SP) [2–4] as well as in the broken symmetry phase (BSP) [9] in one-space, one-time dimension are recapitulated in the next section [2–4].

Quantization of field theory models has also been a challenging problem. In the present work we consider a consistent Hamiltonian [10] and Becchi–

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Rouet–Stora–Tyutin (BRST) [11–14] quantization of the AHM in $(1 + 1)$ dimension with specific gauge choices [11–14].

Further, in the usual Hamiltonian formulation of a gauge-invariant theory under some gauge-fixing conditions (GFC) one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a gauge-invariant theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation [11–14]. In the BRST formulation of a gauge-invariant theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called BRST symmetry. For this one enlarges the Hilbert space of the gauge-invariant theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and anti-ghost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field [11–14]. In the BRST formulation, one thus embeds a gauge-invariant theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as anti-BRST charge operator $\overline{Q}$. The new symmetry of the quantum system (the BRST symmetry) that replaces the gauge invariance is maintained (even under the gauge fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant state yields a theory that is isomorphic to the original gauge-invarinat theory.

The Hamiltonian and BRST formulations of the AHM in the SP [2–4] have been studied in ref. 2. In the present work the model is studied in the BSP [9]. After a brief recapitulation of the basics of the AHM (in the SP as well as in the BSP) in the next section, its Hamiltonian formulation in the BSP is considered in Section 3 and its BRST formulation also in the BSP is studied in Section 4.

2. SOME BASICS OF THE AHM: A RECAPITULATION

2.1. AHM in the Symmetry Phase

The two-dimensional AHM in the symmetry phase is defined by the action [2–4]

$$S = \int \mathscr{L}(\Phi, \Phi^*, A^\mu)\, d^2x \tag{2.1a}$$

where

$$\mathscr{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + (\tilde{D}_\mu \Phi^*)(D^\mu \Phi) - V(|\Phi|^2) \tag{2.1b}$$

$$V(|\Phi|^2) = \alpha_0 + \alpha_2 |\Phi|^2 + \alpha_4 |\Phi|^4 \tag{2.1c}$$

$$= \lambda(|\Phi|^2 - \Phi_0^2)^2; \qquad \Phi_0 \neq 0 \tag{2.1d}$$

$$D_\mu = (\partial_\mu + ieA_\mu); \qquad \tilde{D}_\mu = (\partial_\mu - ieA_\mu) \tag{2.1e}$$

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.1f}$$

$$g^{\mu\nu} := \text{diag}\,(+1, -1); \qquad \mu, \nu = 0, 1 \tag{2.1g}$$

The model is known to possess stable, time-independent (i.e., static), classical solutions called topological solitons of the vortex type [1–4, 7].

In a quantum theory of the kind that we are considering here, for a specific form of the Higgs potential which admits static solutions, in general, one could have *two* degenerate minima, a symmetry-breaking minimum and a symmetry-preserving minimum, and correspondingly the theory could have two types of classical solutions, topological vortices with quantized magnetic flux as we have in the Ginsburg–Landau model, where it is possible to define a conserved topological current and a corresponding topological charge which is quantized and is related to the topological quantum number called as the winding number, and as another type of classical solution, nontopological solitons with nonvanishing, but not necessarily quantized magnetic flux [2–4, 7].

The main new result of such studies is the identification of the Ginsburg–Landau theory with the static solution of the Higgs type of Lagrangian [1–7].

Further, in the present AHM, considered with a Higgs potential in the form of a double-well potential with $\Phi_0 \neq 0$, the spontaneous symmetry breaking (SSB) takes place due to the noninvariance of the lowest (ground) state of the system (because $\Phi_0 \neq 0$) under the operation of the local $U(1)$ symmetry. Also, the symmetry that is broken is still a symmetry of the system and it is manifested in a manner other than the invariance of the lowest or ground state (Φ_0) of the system. However, no Goldstone boson occurs here and instead the gauge field acquires a mass through some kind of Higgs mechanism and the symmetry is manifested in the Higgs mode.

In general one can keep the Higgs potential rather general, i.e., without making any specific choice for the parameters of the potential except that they are chosen such that the potential remains a double-well potential with

$\Phi_0 \neq 0$. For further details we refer to refs. 2–4 and 7 and references therein. The Hamiltonian and BRST formulations of the AHM in the SP have been studied in ref. 2. In the present work the model is studied in the BSP.

2.2. AHM in the Broken Symmetry Phase (BSP)

In the present work we study the AHM in the BSP [9] of the complex matter field Φ [$\equiv \Phi(x, t)$]. For this purpose, for the complex matter field Φ we take

$$\Phi = \Phi_0 \exp[i\phi]; \qquad \Phi_0 \neq 0 \tag{2.2}$$

Here ϕ [$\equiv \phi(x, t)$] is the phase of the complex scalar field Φ. The action of the theory [2–4] in the BSP [9] then becomes

$$S = \int \mathscr{L}\, dx\, dt \tag{2.3a}$$

$$\mathscr{L} := \left[\frac{-1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \phi + eA_\mu)(\partial^\mu \phi + eA^\mu)\right] \tag{2.3b}$$

It is important to notice that the vector gauge boson A^μ becomes massive in the BSP. This mass generation of the vector gauge boson takes place perhaps through a mechanism similar to the Higgs mechanism [9]. The phase ϕ carries the charge degree of freedom of Φ and is in fact akin to the Goldstone boson and is to be treated as a dynamical field [9]. Also, the ground state in the BSP is not rotational invariant. Such studies of the theory in the broken-symmetry (superfluid) state could be relevant for effective theories in condensed matter as the action of the theory describes the low-lying excitations in the BSP [9]. In the present work we study the Hamiltonian and BRST formulations of the theory described by the action (2.3) (in the next two sections, respectively).

3. HAMILTONIAN FORMULATION

For considering the Hamiltonian formulation of the AHM in the BSP in the instant form (i.e., on the hyperplanes $x^0 = \text{const}$), we express the action of the theory (2.3) in component form, which in $(1 + 1)$ dimensions reads [2–4, 7]

$$S = \int \mathscr{L}\, dx\, dt \tag{3.1a}$$

$$\mathscr{L} = \left[\frac{1}{2e^2}[\partial_1 A_0 - \partial_0 A_1]^2 + \frac{1}{2}[(\partial_0\phi)^2 - (\partial_1\phi)^2]\right.$$

$$+ \, e[A_0(\partial_0\phi) - A_1(\partial_1\phi)] + \frac{1}{2}\, e^2(A_0^2 - A_1^2)\Bigg] \tag{3.1b}$$

$$F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu), \qquad g^{\mu\nu} := \mathrm{diag}(+1, -1); \qquad \mu, \nu = 0, 1 \tag{3.1c}$$

In the following, we consider the Hamiltonian formulation of the theory described by the action (3.1). The Euler–Lagrange field equations of motion of the theory obtained from (3.1) are

$$\Big[- \partial_\mu \partial^\mu \phi - e\partial_\mu A^\mu \phi\Big] = 0 \tag{3.2a}$$

$$\left[- e(\partial_1\phi) - e^2 A_1 + \frac{1}{e^2}\, \partial_0 F_{10}\right] = 0 \tag{3.2b}$$

$$\left[e(\partial_0\phi) + e^2 A_0 + \frac{1}{e^2}\, \partial_1 F_{01}\right] = 0 \tag{3.2c}$$

The canonical momenta obtained from (3.1) are

$$\pi := \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi + eA_0 \tag{3.3a}$$

$$\Pi^0 := \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0 \tag{3.3b}$$

$$E(:= \Pi^1) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_1)} = \frac{-1}{e^2}\, (\partial_1 A_0 - \partial_0 A_1) \tag{3.3c}$$

Here π, Π^0, and E $(:= \Pi^1)$ are the momenta canonically conjugate respectively to ϕ, A_0, and A_1. Equations (3.3) imply that the theory possesses one primary constraint:

$$\chi_1 = \Pi_0 \approx 0 \tag{3.4}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}$ of (3.1b) is

$$\mathcal{H}_c := [\pi(\partial_0\phi) + \Pi_0(\partial_0 A_0) + E(\partial_0 A_1) - \mathcal{L}] \tag{3.5a}$$

$$= \Bigg[\frac{1}{2}\, (\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2)$$

$$+ \, E\partial_1 A_0 - e\pi A_0\Bigg] \tag{3.5b}$$

After including the primary constraint χ_1 in the canonical Hamiltonian density

$\mathcal{H}_c$ with the help of the Lagrange multiplier field u, the total Hamiltonian density $\mathcal{H}_T$ can be written as

$$\mathcal{H}_T := \mathcal{H}_c + \Pi_0 u \tag{3.6}$$

The Hamiltonian equations obtained from the total Hamiltonian

$$H_T = \int \mathcal{H}_T \, dx \tag{3.7}$$

are

$$\partial_0 \phi = \frac{\partial H_T}{\partial \pi} = [\pi - eA_0] \tag{3.8a}$$

$$-\partial_0 \pi = \frac{\partial H_T}{\partial \phi} = [-\partial_1 \partial_1 \phi - e(\partial_1 A_1)] \tag{3.8b}$$

$$\partial_0 A_0 = \frac{\partial H_T}{\partial \Pi_0} = u \tag{3.8c}$$

$$-\partial_0 \Pi^0 = \frac{\partial H_T}{\partial A_0} = [-\partial_1 E - e\pi] \tag{3.8d}$$

$$\partial_0 A_1 = \frac{\partial H_T}{\partial E} = [e^2 E + \partial_1 A_0] \tag{3.8e}$$

$$-\partial_0 E = \frac{\partial H_T}{\partial A_1} = [e^2 A_1 + e\partial_1 \phi] \tag{3.8f}$$

$$\partial_0 u = \frac{\partial H_T}{\partial \Pi_u} = 0 \tag{3.8g}$$

$$-\partial_0 \Pi_u = \frac{\partial H_T}{\partial u} = \Pi_0 \tag{3.8h}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. For the Poission bracket $\{\cdot,\cdot\}_{\mathrm{p}}$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_{\mathrm{p}} := \int dz \sum_{\alpha} \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \tag{3.9}$$

Demanding that the primary constraint χ_1 be preserved in the course of time, one obtains the secondary Gauss-law constraint of the theory as

$$\chi_2 := \{\chi_1, \mathcal{H}_T\}_p = [\partial_1 E + e\pi] \approx 0 \tag{3.10}$$

The preservation of χ_2 for all times does not give rise to any further constraints. The theory is thus seen to possess only two constraints χ_1 and χ_2:

$$\chi_1 = \Pi_0 \approx 0; \qquad \chi_2 = [\partial_1 E + e\pi] \approx 0 \tag{3.11}$$

Further, the matrix of the Poisson brackets of the constraints χ_i is seen to be a null matrix, implying that the set of constraints χ_i is first class and that the theory described by (3.1) is a gauge-invariant theory. The action of the theory S(3.1) is, in fact, seen to be invariant under the local-vector gauge transformations (LVGT):

$$\delta A_0 = -\partial_0\beta, \qquad \delta A_1 = -\partial_1\beta, \qquad \delta\phi = e\beta, \qquad \delta u = -\partial_0\partial_0\beta \tag{3.12a}$$

$$\delta\pi = \delta\Pi_u = \delta\Pi_0 = \delta E = 0 \tag{3.12b}$$

where $\beta \equiv \beta(x, t)$ is an arbitrary function of its arguments. The generator of the LVGT is the charge operator of the theory:

$$J^0 = \int j^0\, dx = \int dx \left[e\beta(\partial_0\phi + eA_0) + \frac{1}{e^2}(\partial_1\beta)(\partial_1 A_0 - \partial_0 A_1) \right] \tag{3.13}$$

The current operator of the theory is

$$J^1 = \int j^1\, dx = \int dx \left[-e\beta(\partial_1\phi + eA_1) - \frac{1}{e^2}(\partial_0\beta)(\partial_1 A_0 - \partial_0 A_1) \right] \tag{3.14}$$

The divergence of the vector-current density, namely, $\partial_\mu j^\mu$ is therefore seen to vanish, so that

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_1 j^1 = 0 \tag{3.15}$$

implying that the theory possesses (at the classical level) a local vector-gauge symmetry (LVGS).

In order to quantize the theory using Dirac's procedure we convert the set of first-class constraints of the theory χ_i into a set of second-class constraints by imposing, arbitrarily, some additional constraints on the system called gauge-fixing conditions or the gauge constraints. For this purpose, for the present theory, we could choose, for example, the set of gauge-fixing conditions (A) $\rho_1 = A_0 = 0$ and $\rho_2 = A_1 = 0$ and (B) $\psi_1 = A_0 = 0$ and $\psi_2 = \partial_1 A_1 = 0$; corresponding to these choice of the gauge-fixing conditions, we have the following two sets of constraints under which the quantization of the theory could be studied:

Set A:

$$\xi_1 = \chi_1 = \Pi_0 \approx 0 \tag{3.16a}$$

$$\xi_2 = \chi_2 = [\partial_1 E + e\pi] \approx 0 \tag{3.16b}$$

$$\xi_3 = \rho_1 = A_0 \approx 0 \tag{3.16c}$$

$$\xi_4 = \rho_2 = A_1 \approx 0 \tag{3.16d}$$

Set B:

$$\eta_1 = \chi_1 = \Pi_0 \approx 0 \tag{3.17a}$$

$$\eta_2 = \chi_2 = [\partial_1 E + e\pi] \approx 0 \tag{3.17b}$$

$$\eta_3 = \psi_1 = A_0 \approx 0 \tag{3.17c}$$

$$\eta_4 = \psi_2 = \partial_1 A_1 \approx 0 \tag{3.17d}$$

The matrices of the Poisson brackets among the set of constraints ξ_i and η_i are now seen to be nonsingular (and therefore invertible) and are omitted here for the sake of bravity.

The Dirac bracket $\{\cdot, \cdot\}_D$ of the two functions A and B is defined as [10]

$$\{A, B\}_D = \{A, B\}_p - \int\int dw\, dz \sum_{\alpha,\beta} [\{A, \Gamma_\alpha(w)\}_p [\Delta^{-1}_{\alpha\beta}(w, z)] \times \{\Gamma_\beta(z), B\}_p] \tag{3.18}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(w, z)$ $[=\{\Gamma_\alpha(w), \Gamma_\beta(z)\}_p]$ is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A, B\}_D \to (-i)[A, B], \qquad i = \sqrt{-1} \tag{3.19}$$

Finally, the nonvanishing equal-time commutators of the theory in case A, i.e., in the gauge $A_0 = 0$ and $A_1 = 0$, are obtained as [12–14]

$$[\phi(x, t), \pi(y, t)] = i\delta(x - y) \tag{3.20a}$$

$$[A_1(x, t), E(y, t)] = 2i\delta(x - y) \tag{3.20b}$$

$$[\phi(x, t), E(y, t)] = -\frac{1}{2}\, ie\, \epsilon(x - y) \tag{3.20c}$$

where $\epsilon(x - y)$ is a step function defined as

$$\epsilon(x - y) := \begin{cases} +1, & (x - y) > 0 \\ -1, & (x - y) < 0 \end{cases} \tag{3.21}$$

The nonvanishing equal-time commutators of the theory in case B, i.e., in the gauge $A_0 = 0$ and $\partial_1 A_1 = 0$, are seen to be identical with those of case A, as they should, and are also given by (3.20). This is not surprising in view of the fact that the gauges $A_1 = 0$ and $\partial_1 A_1 = 0$ conceptually mean the same thing.

For later use, for considering the BRST formulation of the theory, we convert the total Hamiltonian density into the first-order Lagrangian density $\mathscr{L}_{IO}$:

$$\mathscr{L}_{IO} := [\pi(\partial_0\phi) + \Pi_0(\partial_0 A_0) + E(\partial_0 A_1) + \Pi_u(\partial_0 u) - \mathscr{H}_T]$$
$$= \left[-\frac{1}{2}(\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2) + \pi(\partial_0\phi) + E(\partial_0 A_1)\right] \tag{3.22}$$

In the above equation, the term $\Pi_0(\partial_0 A_0 - u)$ drops out in view of Hamilton's equations.

4. BRST FORMULATION

4.1. BRST Invariance

For the BRST formulation of the model, we rewrite the theory as a quantum system that possess the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant theory and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics; we then introduce new anticommuting variable c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b such that [11–14]

$$\hat{\delta}\phi = ec; \qquad \hat{\delta}A_1 = -\partial_1 c; \qquad \hat{\delta}A_0 = -\partial_0 c \tag{4.1a}$$
$$\hat{\delta}\pi = \hat{\delta}E = \hat{\delta}\Pi_0 = \hat{\delta}\Pi_u = 0; \qquad \hat{\delta}u = -\partial_0\partial_0 c \tag{4.1b}$$
$$\hat{\delta}c = 0; \qquad \hat{\delta}\bar{c} = b; \qquad \hat{\delta}b = 0 \tag{4.1c}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\pi, E, \Pi_0, \Pi_u, p_b, \Pi_c, \Pi_{\bar{c}}, \phi, A_1, A_0, u, b, c, \bar{c})$ such that $\hat{\delta}f = 0$.

4.2. Gauge Fixing in the BRST Formulism

Performing gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{IO}$ a trivial BRST-invariant function [11–14]. We thus write

$$\mathcal{L}_{\mathrm{BRST}} = \left\{ -\frac{1}{2}\left[\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2\right] + \pi(\partial_0\phi) + E(\partial_0A_1) + \hat{\delta}\left[\overline{c}\left(-\partial_0A_0 + \frac{1}{e}\phi - \frac{1}{2}b\right)\right]\right\} \quad (4.2)$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, the above equation can be written as

$$\mathcal{L}_{\mathrm{BRST}} = \left\{ -\frac{1}{2}\left[\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2\right] + \pi(\partial_0\phi) + E(\partial_0A_1) + b\left(-\partial_0A_0 + \frac{1}{e}\phi\right) - \frac{1}{2}b^2 + (\partial_0\overline{c})(\partial_0c) - \overline{c}\,c\right\} \quad (4.3)$$

Proceeding classically, we see that the Euler–Lagrange equation for b reads

$$-b = \left(\partial_0A_0 - \frac{1}{e}\phi\right) \quad (4.4)$$

The requirement $\hat{\delta}b = 0$ then implies

$$-\hat{\delta}b = \left[\hat{\delta}(\partial_0A_0) - \frac{1}{e}\hat{\delta}\phi\right] \quad (4.5)$$

which in turn implies

$$-\partial_0\partial_0c = c \quad (4.6)$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{\mathrm{BRST}}$ with respect to $\overline{c}$. In introducing momenta, one has to be careful in defining those for the fermionic variables. We thus define the bosonic momenta in the usual manner so that

$$\Pi_0 := \frac{\partial}{\partial(\partial_0A_0)}\mathcal{L}_{\mathrm{BRST}} = -b \quad (4.7)$$

but for the fermionic momenta with directional derivatives we set

$$\Pi_c := \mathcal{L}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial(\partial_0c)} = \mathring{\overline{c}}; \qquad \Pi_{\overline{c}} := \frac{\overrightarrow{\partial}}{\partial(\partial_0\overline{c})}\mathcal{L}_{\mathrm{BRST}} = \mathring{c} \quad (4.8)$$

implying that the variable canonically conjugate to c is $(\partial_0\overline{c})$ and the variable conjugate to $\bar{c}$ is $(\partial_0 c)$. For writing the Hamiltonian density from the Lagrangian density in the usual manner we remember that the former has to be Hermitian, so that

$$\begin{aligned}\mathcal{H}_{\mathrm{BRST}} &= [\pi\mathring{\phi} + \Pi_0\mathring{A}_0 + E\mathring{A}_1 + \Pi_u\mathring{u} + \Pi_c\mathring{c} + \mathring{\bar{c}}\Pi_{\bar{c}} - \mathcal{L}_{\mathrm{BRST}}] \\ &= \Bigg[\frac{1}{2}(\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2) \\ &\quad + \frac{1}{e}\Pi_0\phi + \frac{1}{2}\Pi_0^2 + \Pi_c\Pi_{\bar{c}} + \bar{c}c\Bigg] \end{aligned} \tag{4.9}$$

We can check the consistency of (4.8) and (4.9) by looking at Hamilton's equations for the fermionic variables, i.e.,

$$\partial_0 c = \frac{\vec{\partial}}{\partial\Pi_c}\mathcal{H}_{\mathrm{BRST}}, \qquad \partial_0\bar{c} = \mathcal{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} \tag{4.10}$$

Thus we see that

$$\partial_0 c = \frac{\vec{\partial}}{\partial\Pi_c}\mathcal{H}_{\mathrm{BRST}} = \Pi_{\bar{c}}, \qquad \partial_0\bar{c} = \mathcal{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} = \Pi_c \tag{4.11}$$

is in agreement with (4.8) For the operators c, $\bar{c}$, $\partial_0 c$, and $\partial_0\bar{c}$ one needs to satisfy the anticommutation relations of $\partial_0 c$ with $\bar{c}$ or of $\partial_0\bar{c}$ with c, but not of c, with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0; \qquad \partial_0\{\bar{c}, c\} = 0 \tag{4.12a}$$

$$\{\partial_0\bar{c}, c\} = (-1)\{\partial_0 c, \bar{c}\} \tag{4.12b}$$

where $\{\cdot, \cdot\}$ means an anticommutator. We thus see that the anticommulators in (4.12b) are nontrivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation [11–14]

$$[c, \mathcal{H}_{\mathrm{BRST}}] = i\partial_0\, c \tag{4.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathcal{H}_{\mathrm{BRST}}] = \{\partial_0\bar{c}, c\}\partial_0 c \tag{4.14}$$

Equations (4.12)–(4.14) then imply

$$\{\partial_0\,\bar{c}, c\} = (-1)\{\partial_0 c, \bar{c}\} = i \tag{4.15}$$

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory [3, 10, 11].

4.3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\phi, Q] = -ec; \qquad [A_1, Q] = -\partial_1 c; \qquad [A_0, Q] = \partial_0 c \tag{4.16a}$$

$$\{\bar{c}, Q\} = -\Pi_0; \qquad \{\partial_0 \bar{c}, Q\} = [-\partial_1 E - e\pi] \tag{4.16b}$$

All other commutators and anticommutators involving Q vanish. The BRST charge operator of the present theory can be written as

$$Q = \int dx\, [ic(\partial_1 E + e\pi) - i(\partial_0 c)\Pi_0] \tag{4.17}$$

This equation implies that the set of states satisfying the conditions

$$\Pi_0 |\psi\rangle = 0 \tag{4.18a}$$

$$[\partial_1 E + e\pi] |\psi\rangle = 0 \tag{4.18b}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i,e., it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory, we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (4.6). The solution of Eq. (4.6) gives the Heisenberg operator $c(t)$ [and correspondingly $\bar{c}(t)$] as

$$c(t) = e^{it}B + e^{-it}D; \qquad \bar{c}(t) = e^{-it}B^\dagger + e^{it}D^\dagger \tag{4.19}$$

which at time $t = 0$ imply

$$c \equiv c(0) = B + D; \qquad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger \tag{4.20a}$$

$$\mathring{c} \equiv \mathring{c}(0) = i(B - D); \qquad \mathring{\bar{c}} \equiv \mathring{\bar{c}}(0) = -i(B^\dagger - D^\dagger) \tag{4.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\mathring{\bar{c}}, \mathring{c}\} = 0 \tag{4.21a}$$

$$\{\mathring{\bar{c}}, c\} = i = -\{\mathring{c}, \bar{c}\} \tag{4.21b}$$

we now obtain the equations

$$B^2 + \{B, D\} + D^2 = B^{\dagger 2} + \{B^\dagger, D^\dagger\} + D^{\dagger 2} = 0 \tag{4.22a}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{B^\dagger, D\} = 0 \tag{4.22b}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{B^\dagger, D\} = 0 \tag{4.22c}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -1 \tag{4.22d}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} + \{B, D^\dagger\} - \{D, B^\dagger\} = -1 \tag{4.22e}$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \tag{4.23a}$$

$$\{B, D\} = \{B^\dagger, D\} = \{B, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0 \tag{4.23b}$$

$$\{B^\dagger, B\} = -\frac{1}{2}; \qquad \{D^\dagger, D\} = \frac{1}{2} \tag{4.23c}$$

We now let $|0\,\rangle$ denote the fermionic vacuum for which

$$B|0\rangle = D|0\rangle = 0 \tag{4.24}$$

Defining $|0\rangle$ to have norm one, (4.23c) implies

$$\langle 0|BB^\dagger|0\rangle = -\frac{1}{2}; \qquad \langle 0|DD^\dagger|0\rangle = +\frac{1}{2} \tag{4.25}$$

so that

$$B^\dagger|0\rangle \neq 0; \qquad D^\dagger|0\rangle \neq 0 \tag{4.26}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\text{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators,

$$\mathcal{H}_{\text{BRST}} = \left[\frac{1}{2}(\pi^2 + e^2E^2 + (\partial_1\phi)^2 + 2eA_1(\partial_1\phi) + e^2A_1^2) + \frac{1}{e}\Pi_0\phi + \frac{1}{2}\Pi_0^2 + 2(B^\dagger B + D^\dagger D)\right] \tag{4.27}$$

and the BRST charge operator Q is

$$Q = \int dx\, i\left[B\left(\partial_1 E + e\pi - i\Pi_0\right) + iD\left(\partial_1 E + e\pi + i\Pi_0\right)\right] \tag{4.28}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) holds, but also additional states for which

$$B|\psi\rangle = D|\psi\rangle = 0 \tag{4.29a}$$

$$\Pi_0|\psi\rangle \neq 0 \tag{4.29b}$$

$$[\partial_1 E + e\pi]|\psi\rangle \neq 0 \tag{4.29c}$$

The Hamiltonian is also invariant under the anti-BRST transformation given by

$$\hat{\bar{\delta}}\phi = -e\bar{c}, \qquad \hat{\bar{\delta}}A_0 = \partial_0\bar{c}, \qquad \hat{\bar{\delta}}A_1 = \partial_1\bar{c}, \qquad \hat{\bar{\delta}}u = \partial_0\partial_0 c \tag{4.30a}$$

$$\hat{\bar{\delta}}\pi = \hat{\bar{\delta}}E = \hat{\bar{\delta}}\Pi_0 = \hat{\bar{\delta}}\Pi_u = 0 \tag{4.30b}$$

$$\hat{\bar{\delta}}\bar{c} = 0, \qquad \hat{\bar{\delta}}c = -\,b, \qquad \hat{\bar{\delta}}b = 0 \tag{4.30c}$$

with the generator or anti-BRST charge

$$\begin{aligned}\bar{Q} &= \int dx \left[-i\bar{c}(\partial_1 E + e\pi) + i(\partial_0\bar{c})\Pi_0 \right] \\ &= \int dx \left[-iB^\dagger\left(\partial_1 E + e\pi + i\Pi_0\right) - iD^\dagger\left(\partial_1 E + e\pi - i\Pi_0\right)\right] \end{aligned}\tag{4.31}$$

we also have

$$\partial_0 Q = [Q, H_{\rm BRST}] = 0 \tag{4.32a}$$

$$\partial_0 \bar{Q} = [\bar{Q}, H_{\rm BRST}] = 0 \tag{4.32b}$$

with

$$H_{\rm BRST} = \int dx\, \mathscr{H}_{\rm BRST} \tag{4.32c}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \qquad \text{and} \qquad \bar{Q}|\psi\rangle = 0 \tag{4.33}$$

The states for which (4.18) hold satisfy both of these conditions and, in fact, are the only states satisfying both of these conditions, since, although with (4.23)

$$2(B^\dagger B + D^\dagger D) = -2(BB^\dagger + DD^\dagger) \tag{4.34}$$

there are no states of this operator with $B^\dagger|0\rangle = 0$ and $D^\dagger|0\rangle = 0$ [cf. (4.26)], and hence no free eigenstates of the fermionic part of $H_{\rm BRST}$ which are annihilated by each of B, $B^\dagger$, D, $D^\dagger$. Thus the only states satisfying (4.33) are those satisfying the constraints (3.11).

Further, the states for which (4.18) holds satisfy both the conditions (4.33) and, in fact, are the only states satisfying both of these conditions because in view of (4.21), one cannot have simultaneously c, $\partial_0 c$ and $\bar{c}$, $\partial_0 \bar{c}$ applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.34) are those that satisfy the constraints of the theory (3.11) and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) holds, but also additional states for which (4.29) holds. However, $\overline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which (4.18) holds, simply because $B^\dagger|\psi\rangle \neq 0$ and $D^\dagger|\psi\rangle \neq 0$. Thus in this alternative way we also see that the states satisfying $Q|\psi\rangle = \overline{Q}|\psi\rangle = 0$ [i.e., satisfying (4.33)] are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

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